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Magnetic strings in anti-de Sitter General Relativity

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Abstract

We obtain spacetimes generated by static and spinning magnetic string sources in Einstein Relativity with negative cosmological constant ($\Lambda < 0$). Since the spacetime is asymptotically a cylindrical anti-de Sitter spacetime, we will be able to calculate the mass, momentum, and electric charge of the solutions. We find two families of solutions, one with longitudinal magnetic field and the other with angular magnetic field. The source for the longitudinal magnetic field can be interpreted as composed by a system of two symmetric and superposed electrically charged lines with one of the electrically charged lines being at rest and the other spinning. The angular magnetic field solution can be similarly interpreted as composed by charged lines but now one is at rest and the other has a velocity along the axis. This solution cannot be extended down to the origin.

1 Introduction

1.1 Purpose:

The purpose of this paper is to present asymptotically anti-de Sitter spacetimes generated by static and spinning magnetic sources in Einstein Relativity with negative cosmological constant ($\Lambda < 0$) and with topology different from spherical. These magnetic string sources are obtained in the limit that the string internal structure is restricted to an infinite line source. Since the spacetime is asymptotically anti-de Sitter we will be able to calculate the mass, momentum, and electric charge of the solutions. We find two families of solutions, one with longitudinal magnetic field [the only non-vanishing component of the vector potential is $A_\varphi(r)$] and the other with angular magnetic field [$A_z(r) \neq 0$].

1.2 Black strings:

Static and rotating uncharged solutions of Einstein Relativity with negative cosmological constant and with planar symmetry (planar, cylindrical and toroidal topology) have been found by Lemos [1]. Unlike the zero cosmological constant planar case, these solutions include the presence of black strings (or cylindrical black holes) in the cylindrical model, and of toroidal black holes and black membranes in the toroidal and planar models, respectively. Klemm, Moretti and Vanzo [2] extracted from the general Petrov type-D solution a different rotating toroidal metric of the theory. Topological multi-tori black hole solutions have also been studied [3].

The extension to include the Maxwell field has been done by Zanchin and Lemos [4] who found the static and rotating pure electrically charged black holes that are the electric counterparts of the cylindrical, toroidal and planar black holes found in [1]. The metric with electric charge and zero angular momentum was also discussed by Huang and Liang [5].

Since the discovery of these black holes with cylindrical, toroidal and planar topology, many works have appeared dedicated to the study of their properties. As a test to the cosmic censorship and hoop conjectures, gravitational collapse of these black holes in a background with a negative cosmological constant has been studied by Lemos [6] and Ghosh [7]. The relationship between the topology of the horizon and the topology at infinity was established in [8] for cylindrical, toroidal and planar black holes. The thermodynamics of asymptotically anti-de Sitter spacetimes and, in particular, of toroidal black holes has been studied in [9]. DeBenedictis [10] has studied scalar vacuum polarization effects in cylindrical black holes. The supersymmetry properties of toroidal and cylindrical black holes in anti-de Sitter spacetimes have been obtained in [11] and their quasi-normal modes have been studied by Cardoso and Lemos [12]. A more complete and recent review on this subject and on its various connections can be found in [13].

1.3 Bare (or cosmic) strings:

In this paper we are dealing directly with the issue of spacetimes generated by string sources in four dimensional Einstein theory that are horinzonless and have nontrivial external solutions. A short review of papers treating this subject follows. Levi-Civita [14] and Marder [15] have given static uncharged cylindrically and axially symmetric solutions of Einstein gravity with vanishing cosmological constant. Since then, Vilenkin [16], Ford and Vilenkin [17], Hiscock [18], Gott [19], Harari and Sikivie [20], Cohen and Kaplan [21], Gregory [22] and Banerjee, Banerjee and Sen [23] have found similar static solutions in the context of cosmic string theory. Cosmic strings are topological structures that arise from the possible phase transitions to which the universe might have been subjected to and may play an important role in the formation of primordial structures. These solutions [14]-[23] have in common a geometrical property. They are all horizonless and the corresponding space has a conical geometry, i.e, it is everywhere flat except at the location of the line source. The space can be obtained from the flat space by cutting out a wedge and identifying its edges. The wedge has an opening angle which turns to be proportional to the source mass.

Electromagnetic strings, i.e, horizonless cylindrically and axially symmetric solutions of Einstein-Maxwell gravity have also been obtained. Static electrically charged solutions with $\Lambda = 0$ were found by Mukherji [24] while static magnetic solutions with $\Lambda = 0$ have been constructed by Bonnor [25], Witten [26] and Melvin [27]. Superpositions of these solutions were considered by Safko [28]. In the context of electromagnetic cosmic strings, Witten [29] has shown that there are cosmic strings, known as superconducting cosmic strings, that behave as superconductors and have interesting interactions with astrophysical magnetic fields (see also [30]). These strings can also produce large astronomical magnetic fields [31]. Moss and Poletti [32] studied the gravitational properties of superconducting cosmic strings while non stationary superconducting cosmic strings that reduce, in a certain limit, to the solutions found by Witten [26] were obtained by Gleiser and Tiglio [33]. Superconducting cosmic strings have also been studied in Brans-Dicke theory by Sen [34], and in dilaton gravity by Ferreira, Guimarães and Helayel-Neto [35].

1.4 Corresponding three dimensional solutions:

The relation between cylindrically symmetric four dimensional solutions and spacetimes generated by point sources in three dimensions has been noticed in many works. For example, the cylindrical black hole found in [1] reduces (through dimensional reduction) to a special case of the black holes of a Einstein-Dilaton theory of the Brans-Dicke type [36]. The black holes of a Einstein-Maxwell-Dilaton theory of the Brans-Dicke type [37] contain as a special case the three dimensional counterpart of the electrically charged black hole found in [4]. Now, the study of horizonless spacetime solutions generated by point sources in three dimensions has begun with Staruszkiewicz [38] and Deser, Jackiw and t' Hooft [39] who have studied the three dimensional counterparts of the four dimensional solutions found by Marder [15]. Since then, this issue has been object of many studies. For reviews see [40, 41].

1.5 Plan:

The plan of this article is the following. Section 2 deals with the longitudinal magnetic field solution ($A_\varphi \neq 0$) and section 3 treats the angular magnetic field solution ($A_z \neq 0$). In section 2.1 we set up the action and the field equations. In section 2.2, the static longitudinal solution is found and we analyse in detail the causal and geodesic structure. Angular momentum is added in section 2.3. In section 2.4, we calculate the mass, angular momentum, and electric charge of the longitudinal solutions. In section 2.5, we give a physical interpretation for the source of the longitudinal magnetic field. The angular magnetic solution is discussed in section 3. Finally, in section 4 we present the concluding remarks.

2 Longitudinal magnetic field solution

2.1 Field equations

We are going to work with the Einstein-Hilbert action coupled to electromagnetism in four dimensions with a negative cosmological term

$$S = \frac{1}{8\pi}\int d^4x\sqrt{-g}[R - 2\Lambda - F^{\mu\nu}F_{\mu\nu}], \tag{1}$$

where g is the determinant of the metric, R is the curvature scalar, Λ is the negative cosmological constant and $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ is the Maxwell tensor, with A_μ being the vector potential. We work with units such that $G \equiv 1$ and $c \equiv 1$.

Varying this action with respect to $g^{\mu\nu}$ and $F^{\mu\nu}$ one gets the Einstein and Maxwell equations, respectively

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi T_{\mu\nu}, \tag{2}$$

$$\nabla_\nu F^{\mu\nu} = 0\,, \tag{3}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, ∇ represents the covariant derivative and $T_{\mu\nu} = \frac{1}{4\pi}(g^{\gamma\sigma}F_{\mu\gamma}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F_{\gamma\sigma}F^{\gamma\sigma})$ is the Maxwell energy-momentum tensor.

We want to consider now a spacetime which is both static and rotationally symmetric, implying the existence of a timelike Killing vector $\partial/\partial t$ and a spacelike Killing vector $\partial/\partial\varphi$. We will work with the following ansatz for the metric

$$ds^2 = -\alpha^2 r^2 dt^2 + e^{-2\nu(r)}dr^2 + \frac{e^{2\nu(r)}}{\alpha^2}d\varphi^2 + e^{2\mu(r)}dz^2\,, \tag{4}$$

where the parameter α^2 is, as we shall see, an appropriate constant proportional to the cosmological constant Λ. It is introduced in order to have metric components with dimensionless units and an asymptotically anti-de Sitter spacetime. The motivation for this curious choice for the metric gauge [$g_{tt} \propto -r^2$ and $(g_{rr})^{-1} \propto g_{\varphi\varphi}$] instead of the usual Schwarchild gauge [$(g_{rr})^{-1} = -g_{tt}$ and $g_{\varphi\varphi} = r^2$] comes from the fact that we are looking for magnetic solutions. Indeed, let us first remember that the Schwarzschild gauge is usually an appropriate choice when we are interested on electric solutions. Now, we focus on the well known fact that the electric field is associated with the time component, A_t, of the vector potential while the magnetic field is associated with the angular component A_φ. From the above facts, one can expect that a magnetic solution can be written in a metric gauge in which the components g_{tt} and $g_{\varphi\varphi}$ interchange their roles relatively to those present in the Schwarzschild gauge used to describe electric solutions. This choice will reveal to be a good one to find solutions since we will get a system of differential equations that have a straightforward exact solution. However, as we will see, it is not the good coordinate system to interpret the solutions.

We also assume that the only non-vanishing components of the vector potential are $A_t(r)$ and $A_\varphi(r)$, i.e. ,

$$A = A_t dt + A_\varphi d\varphi \, . \tag{5}$$

This implies that the non-vanishing components of the symmetric Maxwell tensor are F_{tr} and $F_{r\varphi}$. Inserting metric (4) into equation (2) one obtains the following set of equations

$$-\frac{\nu_{,r}}{r} - \frac{\mu_{,r}}{r}(1 + r\nu_{,r}) - \Lambda e^{-2\nu} = -8\pi T_{rr} \, , \tag{6}$$

$$\mu_{,rr} + (\mu_{,r})^2 + \frac{\nu_{,r}}{r} + \frac{\mu_{,r}}{r}(1 + r\nu_{,r}) + \Lambda e^{-2\nu} = 8\pi\alpha^2 e^{-4\nu} T_{\varphi\varphi} \, , \tag{7}$$

$$0 = 8\pi T_{t\varphi} = 2e^{2\nu} F_{tr} F_{\varphi r} \, , \tag{8}$$

where $_{,r}$ denotes a derivative with respect to r. In addition, inserting metric (4) into equation (3) yields

$$\partial_r [r e^{\mu} (F^{tr} + F^{\varphi r})] = 0 \, . \tag{9}$$

2.2 Static longitudinal solution. Analysis of its general structure

2.2.1 Static solution and causal structure

Equations (6)-(9) are valid for a static and rotationally symmetric spacetime. One sees that equation (8) implies that the electric and magnetic fields cannot be simultaneously non-zero, i.e., there is no static dyonic solution. In this work we will consider the magnetically charged case alone ($A_t = 0$, $A_\varphi \neq 0$). For purely electrically charged solutions of the theory see [4]. So, assuming vanishing electric field, one has from Maxwell equation (9) that

$$F^{\varphi r} = \frac{2\chi_{\rm m}}{r} e^{-\mu}, \tag{10}$$

where $\chi_{\rm m}$ is an integration constant which measures the intensity of the magnetic field source. One then has that

$$T_{rr} = \frac{\chi_{\rm m}^2}{2\pi\alpha^2 r^2} e^{-2\nu} e^{-2\mu} \,, \quad T_{\varphi\varphi} = \frac{\chi_{\rm m}^2}{2\pi\alpha^4 r^2} e^{2\nu} e^{-2\mu} \,. \tag{11}$$

Adding equations (6) and (7) one obtains $\mu_{,rr} = -(\mu_{,r})^2$, yielding for the g_{zz} component the solution

$$e^{2\mu} = \alpha^2 r^2 \,, \tag{12}$$

where we have defined $\alpha^2 \equiv -\Lambda/3 > 0$.

The vector potential $A = A_\mu(r) dx^\mu = A_\varphi(r) d\varphi$ with $A_\varphi(r) = \int F_{\varphi r} dr$ is then

$$A = -\frac{2\chi_{\rm m}}{\alpha^3 r} d\varphi \,. \tag{13}$$

Inserting the solutions (11) and (12) into equation (6) we obtain finally the spacetime generated by the static magnetic source

$$\begin{aligned} ds^2 &= -(\alpha r)^2 dt^2 + \frac{dr^2}{(\alpha r)^2 + b(\alpha r)^{-1} - 4\chi_{\rm m}^2 (\alpha^2 r)^{-2}} \\ &\quad + \frac{1}{\alpha^2}[(\alpha r)^2 + b(\alpha r)^{-1} - 4\chi_{\rm m}^2(\alpha^2 r)^{-2}] d\varphi^2 + (\alpha r)^2 dz^2 \,, \end{aligned} \tag{14}$$

where b is a constant of integration related with the mass of the solutions, as will be shown.

In order to study the general structure of solution (14), we first look for curvature singularities. The Kretschmann scalar,

$$R_{\mu\nu\gamma\sigma}R^{\mu\nu\gamma\sigma} = 24\alpha^4\left[1+\frac{b^2}{2(\alpha r)^6}\right] - \frac{4\chi_{\rm m}^2}{\alpha^5 r^7}\left[b - \frac{7\chi_{\rm m}^2}{6\alpha^3 r}\right], \tag{15}$$

diverges at $r=0$ and therefore one might think that there is a curvature singularity located at $r=0$. However, as will be seen below, the true spacetime will never achieve $r=0$.

Now, we look for the existence of horizons and, in particular, we look for the possible presence of magnetically charged black hole solutions. We will conclude that there are no horizons and thus no black holes. The horizons, r_+, are given by the zeros of the Δ function, where $\Delta = (g_{rr})^{-1}$. We find that Δ has one root r_+ given by,

$$r_+ = \frac{b^{\frac{1}{3}}}{2\alpha}\left[\sqrt{\frac{2}{\sqrt{s}} - s} - \sqrt{s}\right], \tag{16}$$

where

$$s = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{4h^2}{3}\right)^3}\right)^{\frac{1}{3}} - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{4h^2}{3}\right)^3}\right)^{\frac{1}{3}}, \tag{17}$$

$$h^2 = \frac{16\chi_{\rm m}^2}{b^{\frac{4}{3}}}. \tag{18}$$

The $\Delta(r)$ function is negative for $r<r_+$ and positive for $r>r_+$. We might then be tempted to say that the solution has an horizon at $r=r_+$ and consequently that we are in the presence of a magnetically charged black hole. However, the above analysis is wrong. In fact, we first notice that the metric components $g_{rr}=\Delta^{-1}$ and $g_{\varphi\varphi}$ are related by $g_{\varphi\varphi} = (\alpha^2 g_{rr})^{-1}$. Then, when g_{rr} becomes negative (which occurs for $r<r_+$) so does $g_{\varphi\varphi}$ and this leads to an apparent change of signature from $+2$ to -2. This strongly indicates [42] that we are using an incorrect extension and that we should choose a different continuation to describe the region $r<r_+$. Besides, we will verify that one can introduce a new coordinate system so that the spacetime is geodesically complete for $r \geq r_+$ [42]. In fact, our next step is to show that we can choose a new coordinate system for which every null or timelike

geodesic starting from an arbitrary point either can be extended to infinite values of the affine parameter along the geodesic or ends on a singularity.

To achieve our aim we introduce the new radial coordinate ρ,

$$\rho^2 = r^2 - r_+^2 \Rightarrow dr^2 = \frac{\rho^2}{\rho^2 + r_+^2} d\rho^2 \,. \tag{19}$$

With this coordinate change the metric Eq. (14) is written as

$$\begin{aligned} ds^2 = -\alpha^2(\rho^2 + r_+^2)dt^2 + \frac{\frac{\rho^2}{(\rho^2+r_+^2)}}{\left[\alpha^2(\rho^2 + r_+^2) + \frac{b}{[\alpha^2(\rho^2+r_+^2)]^{1/2}} - \frac{4\chi_{\rm m}^2}{\alpha^4(\rho^2+r_+^2)}\right]} d\rho^2 \\ + \frac{1}{\alpha^2}\left[\alpha^2(\rho^2 + r_+^2) + \frac{b}{[\alpha^2(\rho^2 + r_+^2)]^{1/2}} - \frac{4\chi_{\rm m}^2}{\alpha^4(\rho^2 + r_+^2)}\right] d\varphi^2 + \alpha^2(\rho^2 + r_+^2)dz^2 \,, \end{aligned} \tag{20}$$

where $0 \leq \rho < \infty$ and $0 \leq \varphi < 2\pi$. The coordinate z can have the range $-\infty < z < \infty$, or can be compactified, $0 \leq z < 2\pi$.

This spacetime has no horizons and no curvature singularity. However, it has a conic geometry and in particular it has a conical singularity at $\rho = 0$. In fact, using a Taylor expansion, we have that in the vicinity of $\rho = 0$ the metric Eq. (20) is written as

$$\begin{aligned} ds^2 \;\; \sim \;\; & -\alpha^2 r_+^2 dt^2 + \frac{1}{\alpha^2 r_+^2} \frac{1}{[1 - (b/2)(\alpha r_+)^{-3} + (4\chi_{\rm m}^2/\alpha^2)(\alpha r_+)^{-4}]} d\rho^2 \\ & + [1 - (b/2)(\alpha r_+)^{-3} + (4\chi_{\rm m}^2/\alpha^2)(\alpha r_+)^{-4}]\rho^2 d\varphi^2 + \alpha^2 r_+^2 dz^2 \,. \end{aligned} \tag{21}$$

Indeed, there is a conical singularity at $\rho = 0$ since

$$\lim_{\rho\to 0} \frac{1}{\rho}\sqrt{\frac{g_{\varphi\varphi}}{g_{\rho\rho}}} \neq 1 \,, \tag{22}$$

i.e, as the radius ρ tends to zero, the limit of the ratio "circumference/radius" is not 2π. The conical singularity can be removed if one identifies the coordinate φ with the period

$$\text{Period}_\varphi = 2\pi \left[\lim_{\rho\to 0} \frac{1}{\rho}\sqrt{\frac{g_{\varphi\varphi}}{g_{\rho\rho}}}\right]^{-1} = 2\pi(1 - 4\mu) \,, \tag{23}$$

with μ given by

$$\mu = \frac{1}{4}\Big[1 - \frac{1}{\alpha r_+ - (b/2)(\alpha r_+)^{-2} + (4\chi_{\rm m}^2/\alpha^2)(\alpha r_+)^{-3}}\Big] \,. \tag{24}$$

From (21)-(24) one concludes that near the origin, $\rho = 0$, metric (20) describes a spacetime which is locally flat but has a conical singularity at $\rho = 0$ with an angle deficit $\delta\varphi = 8\pi\mu$. Since near the origin our metric (21) is identical to the spacetime generated by a cosmic string we can use the procedure of Vilenkin [43] to show that the stress-energy tensor of the string is

$$T_\mu^\nu = (T_t^t, T_x^x, T_y^y, T_z^z) = \mu\delta(x)\delta(y)(-1,0,0,-1) \,, \tag{25}$$

where μ is the mass per unit length of the string defined in (24) and (x, y) are the cartesian coordinates, $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$.

In (20), when one sets $\alpha = 0$ and $\chi_{\rm m} = 0$ one recovers the Levi-Cevita solution [14]. If one sets $\alpha = 0$ one recovers the Witten solution [26] (see also [44]). So the present paper is an extension to include the cosmological constant, rotation and the definition of conserved quantities (see below).

Note also that there are two main distinct properties relatively to the electric charged solutions [4]. First, the electric solutions have black holes, while the magnetic do not. Second, the electric solutions can have cylindrical, toroidal and planar topologies for the 2-space generated by the Killing vectors ∂_φ and ∂_z, whereas the magnetic solutions can only have cylindrical and toroidal topologies, the first case represents an infinite straight magnetic string and the second a closed one.

2.2.2 Geodesic structure

We want to show that the spacetime described by (20) is both null and timelike geodesically complete. The equations governing the geodesics can be derived from the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\frac{\delta}{2} \,, \tag{26}$$

where τ is an affine parameter along the geodesic which, for a timelike geodesic, can be identified with the proper time of the particle along the geodesic. For a null geodesic one has $\delta = 0$ and for a timelike geodesic

$\delta = +1$. From the Euler-Lagrange equations one gets that the generalized momentums associated with the time coordinate, angular coordinate and z-component are constants: $p_t = E$, $p_\varphi = L$, $p_z = P$. The constant E is related to the timelike Killing vector $(\partial/\partial t)^\mu$ which reflects the time translation invariance of the metric, while the constant L is associated to the spacelike Killing vector $(\partial/\partial\varphi)^\mu$ which reflects the invariance of the metric under rotation and the constant P is associated to the spacelike Killing vector $(\partial/\partial z)^\mu$ which reflects the invariance of the metric under a boost in the z direction. Note that since the spacetime is not asymptotically flat, the constants E, L and P cannot be interpreted as the energy, angular momentum and linear momentum at infinity.

From the metric we can derive directly the radial geodesic,

$$\dot{\rho}^2 = -\frac{1}{g_{\rho\rho}}\frac{E^2 g_{\varphi\varphi} + L^2 g_{tt}}{g_{tt}g_{\varphi\varphi}} - \frac{P^2}{g_{\rho\rho}g_{zz}} - \frac{\delta}{g_{\rho\rho}}\,. \tag{27}$$

Now, using the two useful relations $g_{\varphi\varphi} = \rho^2[\alpha^2(\rho^2 + r_+^2)g_{\rho\rho}]^{-1}$ and $g_{tt}g_{\varphi\varphi} = -\rho^2/g_{\rho\rho}$, we can write Eq. (27) as

$$\rho^2\dot{\rho}^2 = \frac{\rho^2}{g_{\rho\rho}}\Big[\frac{E^2 - P^2}{\alpha^2(\rho^2 + r_+^2)} - \delta\Big] - L^2\alpha^2\rho^2\,. \tag{28}$$

Noticing that $1/g_{\rho\rho}$ is always positive for $\rho > 0$ and zero for $\rho = 0$, we conclude the following about the null geodesic motion ($\delta = 0$). (i) if $E^2 > P^2$, null spiraling ($L \neq 0$) particles that start at $\rho = +\infty$ spiral toward a turning point $\rho_{\rm tp} > 0$ and then return back to infinity, (ii) the null particle coming from infinity hits the conical singularity with vanishing velocity if and only if $E^2 > P^2$ and $L = 0$, (iii) whatever the value of L is, for $E^2 < P^2$ there is no possible null geodesic motion, (iv) the same occurs if $E = P$ and $L \neq 0$, (v) if $E = P$ and $L = 0$, whatever the value of ρ is, one has $\dot{\rho} = 0$ and $\dot{\varphi} = -Lg_{tt}g_{\rho\rho}/\rho^2 = 0$ so the null particle moves in a straight line along the z direction.

Now, we analyse the timelike geodesics ($\delta = +1$). Timelike geodesic motion is possible only if the energy and linear momentum of the particle satisfies $E^2 - P^2 > \alpha^2 r_+^2$. In this case, spiraling ($L \neq 0$) timelike particles are bounded between two turning points, a and b, that satisfy $\rho_{\rm tp}^{\rm a} > 0$ and $\rho_{\rm tp}^{\rm b} < \sqrt{(E^2 - P^2)/\alpha^2 - r_+^2}$, with $\rho_{\rm tp}^{\rm b} \geq \rho_{\rm tp}^{\rm a}$. When the timelike particle has no angular momentum ($L = 0$) there is a turning point located at $\rho^{\rm b}_{\rm tp} = \sqrt{(E^2 - P^2)/\alpha^2 - r_+^2}$ and it hits the conical singularity at $\rho = 0$.

Hence, we confirm that the spacetime described by Eq. (20) is both timelike and null geodesically complete.

2.3 The general rotating longitudinal solution

Now, we want to endow our spacetime solution with a global rotation, i.e we want to add angular momentum to the spacetime. In order to do so we perform the following rotation boost in the t-φ plane (see e.g. [4, 36, 37, 47])

$$
\begin{aligned}
t &\mapsto \gamma t - \frac{\omega}{\alpha^2}\varphi\,, \\
\varphi &\mapsto \gamma\varphi - \omega t\,,
\end{aligned} \tag{29}
$$

where γ and ω are constant parameters. Substituting (29) into (20) we obtain

$$
\begin{aligned}
ds^2 = &-\Big[\frac{\alpha^2(\rho^2+r_+^2)}{(\gamma^2-\omega^2/\alpha^2)^{-1}} - \frac{\omega^2}{\alpha^2}\frac{b}{[\alpha^2(\rho^2+r_+^2)]^{\frac{1}{2}}} + \frac{\omega^2}{\alpha^2}\frac{4\chi_{\rm m}^2}{\alpha^4(\rho^2+r_+^2)}\Big]dt^2 \\
&-\frac{\gamma\omega}{\alpha^2}\Big[b[\alpha^2(\rho^2+r_+^2)]^{-\frac{1}{2}} - 4\chi_{\rm m}^2[\alpha^4(\rho^2+r_+^2)]^{-1}\Big]2dtd\varphi \\
&+\frac{\frac{\rho^2}{(\rho^2+r_+^2)}}{\Big[\alpha^2(\rho^2+r_+^2) + \frac{b}{[\alpha^2(\rho^2+r_+^2)]^{\frac{1}{2}}} - \frac{4\chi_{\rm m}^2}{\alpha^4(\rho^2+r_+^2)}\Big]}d\rho^2 \\
&+\frac{1}{\alpha^2}\Big[\frac{\alpha^2(\rho^2+r_+^2)}{(\gamma^2-\omega^2/\alpha^2)^{-1}} + \frac{\gamma^2 b}{[\alpha^2(\rho^2+r_+^2)]^{\frac{1}{2}}} - \frac{4\gamma^2\chi_{\rm m}^2}{\alpha^4(\rho^2+r_+^2)}\Big]d\varphi^2 \\
&+\alpha^2(\rho^2+r_+^2)dz^2\,.
\end{aligned} \tag{30}
$$

Inserting transformations (29) into (13) we obtain that the vector potential $A = A_\mu(\rho)dx^\mu$ is now given by

$$
A = -\omega A(\rho)dt + \gamma A(\rho)d\varphi\,, \tag{31}
$$

where $A(\rho) = -[2\chi_{\rm m}/(\alpha^3 r_+)]\ln[2(r_+ + \sqrt{\rho^2+r_+^2})/\rho]$.

We choose $\gamma^2 - \omega^2/\alpha^2 = 1$ because in this way when the angular momentum vanishes ($\omega = 0$) we have $\gamma = 1$ and so we recover the static solution. Solution (30) represents a magnetically charged cylindrical stationary spacetime (a spinning magnetic string) and also solves (1). Transformations (29) generate a new metric because they are not permitted global coordinate transformations [45]. Transformations (29) can be done locally, but not globally. Therefore, the metrics (20) and (30) can be locally mapped into each other but not globally, and as such they are distinct.

2.4 Mass, angular momentum and electric charge of the longitudinal solution

As we shall see the spacetime solutions (20) and (30) are asymptotically anti-de Sitter. This fact allows us to calculate the mass, angular momentum and the electric charge of the static and rotating solutions. To obtain these quantities we apply the formalism of Regge and Teitelboim [46] (see also [36, 37, 47, 49]). We first write the metric (20) in the canonical form involving the lapse function $N^0(\rho)$ and the shift function $N^\varphi(\rho)$

$$ds^2 = -(N^0)^2dt^2 + \frac{d\rho^2}{f^2} + H^2(d\varphi + N^\varphi dt)^2 + W^2dz^2\,, \tag{32}$$

where $f^{-2} = g_{\rho\rho}$, $H^2 = g_{\varphi\varphi}$, $W^2 = g_{zz}$, $H^2N^\varphi = g_{t\varphi}$ and $(N^0)^2 - H^2(N^\varphi)^2 = g_{tt}$. Then, the action can be written in the Hamiltonian form as a function of the energy constraint $\mathcal{H}$, momentum constraint $\mathcal{H}_\varphi$ and Gauss constraint G

$$\begin{aligned} S &= -\int dt d^3x[N^0\mathcal{H} + N^\varphi\mathcal{H}_\varphi + A_tG] + \mathcal{B} \\ &= -\Delta t\int d\rho N\frac{\Delta z}{8}\Big[\frac{128\pi^2}{H^3W} + (f^2)_{,\rho}(HW)_{,\rho} + 2f^2(H_{,\rho}W)_{,\rho} + 2f^2HW_{,\rho\rho} \\ &\quad -2\Lambda HW + \frac{2HW}{f}(E^2+B^2)\Big] \\ &\quad +\Delta t\int d\rho N^\varphi\frac{\Delta z}{8}\Big[(2\pi)_{,\rho} + \frac{4HW}{f}E^\rho B\Big] \\ &\quad +\Delta t\int d\rho A_t\frac{\Delta z}{8}\Big[-\frac{4HW}{f}\partial_\rho E^\rho\Big] + \mathcal{B}\,, \end{aligned} \tag{33}$$

where $N = N^0/f$, $\pi \equiv \pi_\varphi{}^\rho = -\frac{fH^3W(N^\varphi)_{,\rho}}{2N^0}$ (with $\pi^{\rho\varphi}$ being the momentum conjugate to $g_{\rho\varphi}$), E^ρ and B are the electric and magnetic fields and $\mathcal{B}$ is a boundary term. Upon varying the action with respect to $f(\rho)$, $H(\rho)$, $W(\rho)$, $\pi(\rho)$, and $E^\rho(\rho)$ one picks up additional surface terms. Indeed,

$$\begin{aligned}
\delta S &= -\Delta t N \frac{\Delta z}{8}\Big[(HW)_{,\rho}\delta f^2 - (f^2)_{,\rho} W \delta H + 2f^2 W \delta(H_{,\rho}) \\
&\quad -H(f^2)_{,\rho}\delta W + 2Hf^2\delta(W_{,\rho})\Big] \\
&\quad +\Delta t N^\varphi \frac{\Delta z}{4}\delta\pi + \Delta t A_t \frac{\Delta z}{8}\Big[-\frac{4HW}{f}\delta E^\rho\Big] + \delta\mathcal{B} \\
&\quad +(\text{terms vanishing when the equations of motion hold}). \qquad (34)
\end{aligned}$$

In order that the Hamilton's equations are satisfied, the boundary term $\mathcal{B}$ has to be adjusted so that it cancels the above additional surface terms. More specifically one has

$$\delta\mathcal{B} = -\Delta t N \delta\bar{M} + \Delta t N^\varphi \delta\bar{J} + \Delta t A_t \delta\bar{Q}_{\rm e}\,, \qquad (35)$$

where one identifies $\bar{M}$ as the mass, $\bar{J}$ as the angular momentum and $\bar{Q}_{\rm e}$ as the electric charge since they are the terms conjugate to the asymptotic values of N, N^φ and A_t, respectively.

To determine the mass, angular momentum and the electric charge of the solutions one must take the spacetime that we have obtained and subtract the background reference spacetime contribution, i.e., we choose the energy zero point in such a way that the mass, angular momentum and charge vanish when the matter is not present.

Now, note that spacetime (30) has an asymptotic metric given by

$$-\Big(\gamma^2 - \frac{\omega^2}{\alpha^2}\Big)\alpha^2\rho^2 dt^2 + \frac{d\rho^2}{\alpha^2\rho^2} + \Big(\gamma^2 - \frac{\omega^2}{\alpha^2}\Big)\rho^2 d\varphi^2 + \alpha^2\rho^2 dz^2\,, \qquad (36)$$

where $\gamma^2 - \omega^2/\alpha^2 = 1$ so, it is asymptotically a cylindrical anti-de Sitter spacetime (we can rescale the coordinates r and z so that the usual form of the anti-de Sitter spacetime becomes apparent). The cylindrical anti-de Sitter spacetime is also the background reference spacetime, since the metric (30) reduces to (36) if the matter is not present ($b = 0$ and $\chi_{\rm m} = 0$). Taking the subtraction of the background reference spacetime into account and noting that $W - W_{\rm ref} = 0$ and that $W_{,\rho} - W^{\rm ref}_{,\rho} = 0$ we have that the mass, angular momentum and electric charge are given by

$$\bar{M} = \frac{\Delta z}{8}\Big[-(HW)_{,\rho}(f^2 - f^2_{\rm ref}) + (f^2)_{,\rho}W(H - H_{\rm ref}) - 2f^2W(H_{,\rho} - H^{\rm ref}_{,\rho})\Big] , \tag{37}$$

$$\bar{J} = -\Delta z(\pi - \pi_{\rm ref})/4 , \tag{38}$$

$$\bar{Q}_{\rm e} = \frac{\Delta z}{2}\frac{HW}{f}(E^\rho - E^\rho_{\rm ref}) . \tag{39}$$

Then, we finally have that the mass per unit length ($\bar{M}/\Delta z$) and the angular momentum per unit length ($\bar{J}/\Delta z$) are (after taking the asymptotic limit, $\rho \to +\infty$)

$$M = \frac{b}{8}\Big[\gamma^2 + 2\frac{\omega^2}{\alpha^2}\Big] + {\rm Div}_{\rm M}(\chi_{\rm m}, \rho) = M_0 + {\rm Div}_{\rm M}(\chi_{\rm m}, \rho) , \tag{40}$$

$$J = \frac{3b}{8}\frac{\gamma\omega}{\alpha^2} + {\rm Div}_{\rm J}(\chi_{\rm m}, \rho) , \tag{41}$$

where ${\rm Div}_{\rm M}(\chi_{\rm m}, \rho)$ and ${\rm Div}_{\rm M}(\chi_{\rm m}, \rho)$ are terms proportional to the magnetic source $\chi_{\rm m}$ that diverge as $\rho \to +\infty$. The presence of these kind of divergences in the mass is a usual feature present in charged solutions. They can be found for example on the electrically charged point source solution in 3D gravity [48], in the electrically charged BTZ black hole [49] and in the electrically charged solutions of three-dimensional Brans-Dicke action [37]. Following [48, 49] (see also [37]) these divergences can be treated as follows. One considers a boundary of large radius ρ_0 involving the system. Then, one sums and subtracts ${\rm Div}_{\rm M}(\chi_{\rm m}, \rho_0)$ to (40) so that the mass per unit length (40) is now written as

$$M = M(\rho_0) + [{\rm Div}_{\rm M}(\chi_{\rm m}, \rho) - {\rm Div}_{\rm M}(\chi_{\rm m}, \rho_0)] , \tag{42}$$

where $M(\rho_0) = M_0 + {\rm Div}_{\rm M}(\chi_{\rm m}, \rho_0)$, i.e.,

$$M_0 = M(\rho_0) - {\rm Div}_{\rm M}(\chi_{\rm m}, \rho_0) . \tag{43}$$

The term between brackets in (42) vanishes when $\rho \to \rho_0$. Then $M(\rho_0)$ is the energy within the radius ρ_0. The difference between $M(\rho_0)$ and $-M_0$ is $-{\rm Div}_{\rm M}(\chi_{\rm m}, \rho_0)$ which is interpreted as the electromagnetic energy outside ρ_0 apart from an infinite constant which is absorbed in $M(\rho_0)$. The sum (43) is then independent of ρ_0, finite and equal to the total mass. In practice the treatment of the mass divergence amounts to forgetting about ρ_0 and take as zero the asymptotic limit: $\lim \mathrm{Div}_{\mathrm{M}}(\chi_{\mathrm{m}}, \rho) = 0$.

To handle the angular momentum divergence, one first notices that the asymptotic limit of the angular momentum per unit mass (J/M) is either zero or one, so the angular momentum diverges at a rate slower or equal to the rate of the mass divergence. The divergence on the angular momentum can then be treated in a similar way as the mass divergence. So, one can again consider a boundary of large radius ρ_0 involving the system. Following the procedure applied for the mass divergence one concludes that the divergent term $-\mathrm{Div}_{\mathrm{J}}(\chi_{\mathrm{m}}, \rho_0)$ can be interpreted as the electromagnetic angular momentum outside ρ_0 up to an infinite constant that is absorbed in $J(\rho_0)$.

Hence, in practice the treatment of both the mass and angular divergences amounts to forgetting about ρ_0 and take as zero the asymptotic limits: $\lim \mathrm{Div}_{\mathrm{M}}(\chi_{\mathrm{m}}, \rho) = 0$ and $\lim \mathrm{Div}_{\mathrm{J}}(\chi_{\mathrm{m}}, \rho) = 0$ in Eqs. (40)-(41).

Now, we calculate the electric charge of the solutions. To determine the electric field we must consider the projections of the Maxwell field on spatial hypersurfaces. The normal to such hypersurfaces is $n^\nu = (1/N^0, 0, -N^\varphi/N^0, 0)$ and the electric field is given by $E^\mu = g^{\mu\sigma} F_{\sigma\nu} n^\nu$. Then, from (39), the electric charge per unit length ($\bar{Q}_{\mathrm{e}}/\Delta z$) is

$$Q_{\mathrm{e}} = -\frac{4HWf}{N^0}(\partial_\rho A_t - N^\varphi \partial_\rho A_\varphi) = \frac{\omega}{\alpha^2}\chi_{\mathrm{m}} \, . \tag{44}$$

Note that the electric charge is proportional to $\omega\chi_{\mathrm{m}}$. In the next subsection we will propose a physical interpretation for the origin of the magnetic field source and discuss the result obtained in (44).

Now, we want to cast the metric (30) in terms of M, J and Q_{e}. We can use (40) and (41) to solve a quadratic equation for γ^2 and ω^2/α^2. It gives two distinct sets of solutions

$$\gamma^2 = \frac{4M}{b}(2-\Omega) \, , \qquad \frac{\omega^2}{\alpha^2} = \frac{2M}{b}\Omega \, , \tag{45}$$

$$\gamma^2 = \frac{4M}{b}\Omega \, , \qquad \frac{\omega^2}{\alpha^2} = \frac{2M}{b}(2-\Omega) \, , \tag{46}$$

where we have defined a rotating parameter Ω as

$$\Omega \equiv 1 - \sqrt{1 - \frac{8}{9}\frac{J^2\alpha^2}{M^2}} \, . \tag{47}$$

When we take $J = 0$ (which implies $\Omega = 0$), (45) gives $\gamma \neq 0$ and $\omega = 0$ while (46) gives the nonphysical solution $\gamma = 0$ and $\omega \neq 0$ which does not reduce to the static original metric. Therefore we will study the solutions found from (45).

The condition that Ω remains real imposes a restriction on the allowed values of the angular momentum:

$$\alpha^2 J^2 \leq \frac{8}{9} M^2 \, . \tag{48}$$

The parameter Ω ranges between $0 \leq \Omega \leq 1$. The condition $\gamma^2 - \omega^2/\alpha^2 = 1$ fixes the value of b as a function of M, Ω,

$$b \;=\; 2M(4 - 3\Omega) \, . \tag{49}$$

The metric (30) may now be cast in the form

$$\begin{aligned}
&ds^2 = \\
&-\Big[\alpha^2(\rho^2 + r_+^2) - 2M\Omega[\alpha^2(\rho^2 + r_+^2)]^{-1/2} + 4Q_{\rm e}^2[\alpha^2(\rho^2 + r_+^2)]^{-1}\Big] dt^2 \\
&-\frac{8}{3}J\Big[[\alpha^2(\rho^2 + r_+^2)]^{-1/2} - \frac{2Q_{\rm e}^2}{M\Omega}[\alpha^2(\rho^2 + r_+^2)]^{-1}\Big] 2dtd\varphi \\
&+\frac{\frac{\rho^2}{(\rho^2 + r_+^2)}}{\Big[\alpha^2(\rho^2 + r_+^2) + 2M(4-3\Omega)[\alpha^2(\rho^2 + r_+^2)]^{-1/2} - 4\chi_{\rm m}^2[\alpha^4(\rho^2 + r_+^2)]^{-1}\Big]} d\rho^2 \\
&+\frac{1}{\alpha^2}\Big[\alpha^2(\rho^2 + r_+^2) + \frac{4M(2-\Omega)}{[\alpha^2(\rho^2 + r_+^2)]^{1/2}} - \frac{8(2-\Omega)}{4-3\Omega}\frac{\chi_{\rm m}^2}{[\alpha^4(\rho^2 + r_+^2)]}\Big] d\varphi^2 \\
&+\alpha^2(\rho^2 + r_+^2) dz^2 \, .
\end{aligned} \tag{50}$$

The static solution can be obtained by putting $\Omega = 0$ (and thus $J = 0$ and $Q_{\rm e} = 0$) on the above expression [see (29)].

2.5 Physical interpretation of the longitudinal magnetic field source

When we look back to the value of the electric charge per unit length, Eq. (44), we see that it is zero when the angular momentum of the spacetime vanishes, $J = 0 = \omega$. This result, no electric source in the static spacetime case, was expected since we have imposed that the static electric field is zero (F_{21} is the only non-zero component of the Maxwell tensor).

Still missing however, is a physical interpretation for the origin of the magnetic field source. It is quite evident that the magnetic field source is not a 't Hooft-Polyakov monopole since we are working with the Maxwell theory and not with an $SO(3)$ gauge theory spontaneously broken to $U(1)$ by an isotriplet Higgs field. We might then think that the magnetic field is produced by a Dirac line-like monopole. However, this is not also the case since the Dirac monopole with strength $g_{\rm m}$ appears when one breaks the Bianchi identity [50]: $\partial_\nu(\sqrt{-g}\tilde{F}^{\mu\nu}) = 4\pi k^\mu/\sqrt{-g}$, where $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\gamma\sigma}F_{\gamma\sigma}/2$ is the dual of the Maxwell field strength and $k^\mu = \sum g_{\rm m}\delta^3(\vec{x}-\vec{x}_0)\dot{x}^\mu$ is the magnetic current density. But in this work we are clearly dealing with the Maxwell theory which satisfies Maxwell equations and the Bianchi identity

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 4\pi\frac{1}{\sqrt{-g}}j^\mu\,, \tag{51}$$

$$\partial_\mu(\sqrt{-g}\tilde{F}^{\mu\nu}) = 0\,, \tag{52}$$

respectively. We have made use of the fact that the general relativistic current density is $1/\sqrt{-g}$ times the special relativistic current density $j^\mu = \sum q\delta^3(\vec{x}-\vec{x}_0)\dot{x}^\mu$. Hence, from (52) we have to put away the hypothesis of the Dirac monopole.

Following [26] the magnetic field source can then be interpreted as composed by a system of two symmetric and superposed electrically charged lines along the z direction (each with strength q per unit length). One of the electrically charged lines is at rest and the other is spinning around the z direction with an angular velocity $\dot{\varphi}_0$. Clearly, this system produces no electric field since the total electric charge is zero and the magnetic field is produced by the rotating electric current. To confirm our interpretation, we go back to Eq. (51). In our solution, the only non-vanishing component of the Maxwell field is $F^{\varphi\rho}$ which implies that only j^φ is not zero. According to our interpretation one has $j^\varphi = q\delta^2(\vec{x}-\vec{x}_0)\dot{\varphi}$, which one inserts in Eq. (51). Finally, integrating over ρ and φ and introducing $F^{\varphi\rho} = 2\chi_{\rm m}/(\alpha\rho\sqrt{\rho^2+r_+^2})$ we have

$$\chi_{\rm m} \propto q\dot{\varphi}_0 \,. \tag{53}$$

So, the magnetic source strength per unit length, $\chi_{\rm m}$, can be interpreted as an electric charge per unit length times its spinning velocity.

Looking again to the value of the electric charge per unit length, Eq. (44), one sees that after applying the rotation boost in the t-φ plane to endow our initial static spacetime with angular momentum, there appears a net electric charge. This result was once again expected since now, besides the magnetic field along the z direction ($F_{\rho\varphi} \neq 0$), there is also a radial electric field ($F_{t\rho} \neq 0$). A physical interpretation for the appearance of the net electric charge is needed. To do so, we first go back to the static spacetime. According to our interpretation, an observer at rest relative to the source (S) sees a density of charges at rest which is equal to the negative of the density of charges that are rotating. Now, we consider the stationary spacetime and we look to the point of view of an observer (S') that follows the intrinsic rotation of the spacetime, i.e. that rotates with the same angular velocity of the spacetime. The observer S' is moving relatively to the observer S and we know that the charge density of a moving distribution of charges varies as the inverse of the relativistic length (since the density is a charge over an area) when we compare the measurements made by two different frames. So, the two set of charge distributions that had symmetric charge densities in the frame S will not have charge densities with equal magnitude in the frame S'. Hence, they will not cancel each other in the frame S' and a net electric charge appears. This analysis is similar to the one that occurs when one has a cupper wire with an electric current along the wire and we apply a translation Lorentz boost to the wire along its direction: initially, there is only a magnetic field but, after the Lorentz boost, one also has an electric field. The only difference is that in the present situation the Lorentz boost is a rotational one and not a translational one.

3 Angular magnetic field solution

In section 2 we have found a spacetime generated by a magnetic source that produces a longitudinal magnetic field along the z direction. In this section we find a spacetime generated by a magnetic source that produces an angular magnetic field along the φ direction. Following the steps of section 2 but now with the roles of φ and z interchanged, we can write directly the metric and electric potential for the angular magnetic field solution as

$$ds^2 = -\alpha^2(\rho^2 + r_+^2)dt^2 + \frac{\frac{\rho^2}{(\rho^2+r_+^2)}}{\left[\alpha^2(\rho^2 + r_+^2) + \frac{b}{[\alpha^2(\rho^2+r_+^2)]^{1/2}} - \frac{4\chi_{\rm m}^2}{\alpha^2(\rho^2+r_+^2)}\right]}d\rho^2$$
$$+(\rho^2 + r_+^2)d\varphi^2 + \left[\alpha^2(\rho^2 + r_+^2) + \frac{b}{[\alpha^2(\rho^2 + r_+^2)]^{1/2}} - \frac{4\chi_{\rm m}^2}{\alpha^2(\rho^2 + r_+^2)}\right]dz^2 \,, \tag{54}$$

where again $\alpha^2 \equiv -\Lambda/3 > 0$, ρ is a radial coordinate, φ an angular coordinate and z ranges between $-\infty < z < \infty$ (or $0 \leq z < 2\pi$),

$$A = -\frac{2\chi_{\rm m}}{\alpha^3 r_+} \ln\left[\frac{2(r_+ + \sqrt{\rho^2 + r_+^2})}{\rho}\right]dz \,. \tag{55}$$

This spacetime has a mass per unit lenght given by $M = b/8$ and no electric charge. The Kretschmann scalar does not diverge for any ρ so there is no curvature singularity. Spacetime (54) is also free of conical singularities. Besides, if one studies the radial geodesic motion we conclude that these geodesics can pass through $\rho = 0$ (which is free of singulaties) from positive values to negative values of ρ. This shows that in (54) the radial coordinate can take the values $-\infty < \rho < \infty$. This analysis might suggest that we are in the presence of a traversable (since the spacetime has no horizons) wormhole with a throat of dimension r_+. However, in the vicinity of $\rho = 0$, metric (54) is written as

$$\begin{aligned} ds^2 \;\sim\; & -\alpha^2 r_+^2 dt^2 + \frac{1}{\alpha^2 r_+^2}\frac{1}{[1 - (b/2)(\alpha r_+)^{-3} + 4\chi_{\rm m}^2(\alpha r_+)^{-4}]}d\rho^2 \\ & + r_+^2 d\varphi^2 + [1 - (b/2)(\alpha r_+)^{-3} + 4\chi_{\rm m}^2(\alpha r_+)^{-4}]\rho^2 dz^2 \,. \end{aligned} \tag{56}$$

which clearly shows that at $\rho = 0$ the z-direction collapses and therefore we have to abandon the wormhole interpretation.

A physical interpretation for the source of the angular magnetic field can be given. The spacetime has zero electric field and angular magnetic field along the φ direction. The source for the magnetic field could then be interpreted as composed by a system of two symmetric and superposed electrically charged lines along the z direction. One of the electrically charged lines would be at rest and the other would have a velocity along the z direction. Clearly, this system produces no electric field since the total electric charge is zero and the magnetic field is produced by the axial electric current. There is however a great problem with this interpretation if we want to extend the solution down to the origin. Indeed, at $\rho = 0$ there is no physical object to support the proposed source, and in addition, the four-dimensional character of the solution is lost (since near the origin the z direction collapses). In order to have a full solution one has to consider a matter source up to a boundary radius $\rho_{\rm b}$ with an appropriate stress-energy tensor which should generate the exterior solution (54).

To add linear momentum to the spacetime, we perform the following translation boost in the t-z plane: $t \mapsto \gamma t - \frac{\lambda}{\alpha} z$, $z \mapsto \gamma z - \frac{\lambda}{\alpha} t$, where γ and λ are constant parameters that are choosen so that $\gamma^2 - \lambda^2/\alpha^2 = 1$ because in this way when the linear momentum vanishes ($\lambda = 0$) we have $\gamma = 1$ and so we recover the static solution. Contrarily to transformation (29), the above translational boost transformation is permitted since z is not an angular coordinate. Thus boosting the solution (54) through this permitted transformation does not yield a new solution. However, it generates an electric field. For the sake of completeness we write below the boosted solution:

$$
\begin{aligned}
& ds^2 = \\
& -\left[\alpha^2(\rho^2+r_+^2) - 2M\Pi[\alpha^2(\rho^2+r_+^2)]^{-1/2} + 4Q_{\rm e}^2[\alpha^4(\rho^2+r_+^2)]^{-1}\right]dt^2 \\
& -\frac{8}{3}P\left[[\alpha^2(\rho^2+r_+^2)]^{-1/2} - \frac{2Q_{\rm e}^2}{M\Pi}[\alpha^4(\rho^2+r_+^2)]^{-1}\right]2dtdz \\
& +\frac{\frac{\rho^2}{(\rho^2+r_+^2)}}{\left[\alpha^2(\rho^2+r_+^2) + 2M(4-3\Pi)[\alpha^2(\rho^2+r_+^2)]^{-1/2} - 4\chi_{\rm m}^2[\alpha^2(\rho^2+r_+^2)]^{-1}\right]}d\rho^2 \\
& +(\rho^2+r_+^2)d\varphi^2
\end{aligned}
$$

$$+\left[\alpha^2(\rho^2+r_+^2)+\frac{4M(2-\Pi)}{[\alpha^2(\rho^2+r_+^2)]^{1/2}}-\frac{8(2-\Pi)}{4-3\Pi}\frac{\chi_{\rm m}^2}{[\alpha^2(\rho^2+r_+^2)]}\right]dz^2\,, \tag{57}$$

where, the mass per unit length, the linear momentum per unit length and the electric charge per unit length of the boosted solution are $M = b(\gamma^2 + 2\lambda^2/\alpha^2)/8$, $P = 3b\gamma\lambda/(8\alpha)$ and $Q_{\rm e} = \lambda\chi_{\rm m}$. We have defined the parameter Π as $\Pi \equiv 1 - \sqrt{1 - 8P^2/(9M^2)}$ and the condition that Π remains real imposes a restriction on the allowed values of the linear momentum: $P^2 \leq \frac{8}{9}M^2$. The vector potential is given by

$$A = \frac{A(\rho)}{\sqrt{8-6\Pi}}\left[-\sqrt{2\Pi}dt + 2\sqrt{2-\Pi}dz\right]\,, \tag{58}$$

where $A(\rho) = -[2\chi_{\rm m}/(\alpha^3 r_+)]\ln[2(r_+ + \sqrt{\rho^2+r_+^2})/\rho]$. Notice that when one applies the translation boost in the t-z plane to endow our initial static spacetime with linear momentum, there appears a net electric charge proportional to $\lambda\chi_{\rm m}$. The static solution can be obtained by putting $\Pi = 0$ (and thus $P = 0$ and $Q_{\rm e} = 0$) on the above expressions.

4 Conclusions

We have added a Maxwell term to Einstein Relativity with $\Lambda < 0$. For the static spacetime, the electric and magnetic fields cannot be simultaneously non-zero, i.e. there is no static dyonic solution. Pure electrically charged solutions of the theory have been studied in detail in [4].

We have found two families of solutions. One yields a spacetime with longitudinal magnetic field [the only non-vanishing component of the vector potential is $A_\varphi(\rho)$] generated by a static magnetic line source (an infinite straight magnetic string or a closed magnetic string). The corresponding spinning magnetic source that produces in addition a radial electric field was also found. The source for the longitudinal magnetic field solution can be interpreted as composed by a system of two symmetric and superposed electrically charged lines with one of the electrically charged lines being at rest and the other spinning. The other solution gives a spacetime with angular magnetic field [$A_z(\rho) \neq 0$]. This angular magnetic field solution can be similarly interpreted as composed by two electrically charged lines but now one is at rest and the other has a velocity along the axis. This solution cannot be extended down to the origin. The two families of solutions are asymptotically anti-de Sitter which allowed us to find the mass, momentum and electric charge of the solutions.

As we said, there is a relation between cylindrically symmetric four dimensional solutions and spacetimes generated by point sources in three dimensions. Therefore, we expect that three dimensional analogue of the solutions presented in this paper exists. The dimensional reduction of the longitudinal magnetic field solution yields a Brans-Dicke theory (see [41]). On the other hand the dimensional reduction of the angular magnetic field solution yields an effective three dimensional theory which is not a pure Brans-Dicke theory since it has an extra gauge field (see [51]).

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